Reminders of Formulas from single var

Generally

$$\frac{d}{dx} ax^3 = 3ax^2$$

We're thinking of "a" as a const but you could think of it as a var

→ You could call it y

$$\frac{d}{dx}(yx^3) = 3yx^2$$

Still y (or a) is still a const bc we're diff'ing wrt x

But you can diff wrt y:

$$\frac{d}{dy}(yx^3) = x^3$$

→ ie, if you plug in some const for x, then the eqn true

eg $x = -1$

$$\frac{d}{dy}(-y) = -1 = (-1)^3$$

eg $\frac{d}{dy}(8y) = 8$

If you want to emphasize that x is const when you diff wrt y, you could call it "a" and write

$$\frac{d}{dy}(ya^3) = a^3$$

Another Formula

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

equivalently:

$$\frac{d}{dx}(e^{yx}) = \frac{d}{dx}(e^{xy}) = ye^{xy}$$

$$\frac{d}{dy}(e^{xy}) = xe^{xy}$$

In MV calc, when you have multiple vars & diff wrt one of them, we write ∂ instead of d.

So

$$\frac{\partial}{\partial x} e^{xy} = ye^{xy}$$

In general, if $f(x,y)$ is a fcn of 2 vars, then

$\frac{\partial f}{\partial x}$ is what you get if you treat y as a const and take a deriv. wrt x

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{y=100} = \frac{d}{dx} \underbrace{f(x,100)}_{\text{single var fcn}}$$

<u>eg</u> $f(x,y) = x\sin(y) + e^x + y$

$$\frac{\partial f}{\partial x} = \sin(y) + e^x$$

$$\frac{\partial f}{\partial y} = x\cos(y) + 1$$

NOT EQUAL!

$$\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial y}$$

"partial derivatives of f"

<u>In 3 variables</u> $f(x,y,z)$

then we have: $\frac{\partial f}{\partial x}, \frac{\partial f}{d y}, \frac{\partial f}{d z}$

<u>eg</u> $f(x,y,z) = xyz$

$$\frac{\partial f}{\partial x} = yz \qquad \frac{\partial f}{\partial y} = xz \qquad \frac{\partial f}{\partial z} = xy$$

What if we differentiate mult times?

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xz) = z$$

equal!

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(yz) = z$$

<u>True in general</u>

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$ as long as the partial derivatives are continuous

In general, we only work w fcns whose nth derivatives exist and are continuous

Caveat: sometimes $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are def'd but not continuous & various properties fails

$D_x f$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ $\quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Exercise $x \sin y + e^x + y$

$\frac{\partial F}{\partial x} \left(\frac{\partial y}{\partial x} \right.$

$= \cos y$

Properties

- sum rule: $\frac{\partial (f+g)}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$
- scalar mult: $\frac{\partial (aF)}{\partial x} = a \frac{\partial f}{\partial x}$, $a \in \mathbb{R}$

  similarly: $\frac{\partial (yF)}{\partial x} = y \frac{\partial f}{\partial x}$
- product rule: $\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial z} \cdot g + f \cdot \frac{\partial g}{\partial z}$

  (same if we mix up x, y, z)

Let $(a,b) \in D \subseteq \mathbb{R}^2$ and f def'd & diff'able on D.

Consider $g(t) = f(a, b+t)$

this is a one-var fcn

$$\frac{dg}{dt} = \frac{\partial f}{\partial y}(a, b+t)$$

Intuitively Why is $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$?

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{\partial}{\partial x} \left(\frac{f(x, y+h) - f(x,y)}{h} \right) \right)$$

assume we can interchange derivative and lim

$$\frac{d}{dx}\left((a+h)\,x^3\right) = 3(a+h)x^2$$

Notice h & y const wrt x

so

$$\frac{\partial}{\partial x}\left(\frac{f(x, y+h) - f(x,y)}{h} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{f(x, y+h) - f(x,y)}{h} \right)$$

$$= \frac{\frac{\partial}{\partial x}\left(f(x, y+h)\right) - \frac{\partial}{\partial x}\left(f(x,y)\right)}{h}$$

$$= \frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x,y)}{h}$$

so $$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h \to 0} \left(\frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x,y)}{h} \right)$$

set $g = \frac{\partial f}{\partial x}$

$$= \lim_{h \to 0} \left(\frac{g(x, y+h) - g(x,y)}{h} \right)$$

$$= \frac{\partial g}{\partial x}$$

Tom Apostol Calc II proves everything rigorously

$$= \frac{\partial(\partial f / \partial x)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

2.3 Tanget Planes

Reminder on tangent lines

$$y = f(x)$$

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\Rightarrow f(x_1) - f(x_0) \approx (x_1 - x_0)\frac{df}{dx}(x_0)$$

$$\Rightarrow \boxed{f(x_1) \simeq f(x_0) + \frac{df}{dx}(x_0) \cdot \Delta x}$$

"linear appoximation"

bc - if fix x_0 & let x_1 vary, then:

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$f(x_0) + \Delta x \cdot \frac{df}{dx}(x_0)$

$= f(x_0) + \frac{df}{dx}(x_0) \cdot (x_1 - x_0)$

is a linear func of x_1 that's "approx" $f(x_1)$

Approximation is best when x_1 is close to x_0

In other words, the line:

$$y = f(x_0) + \frac{df}{dx}(x_0) \cdot (x_1 - x_0)$$

where x_0 is fixed & x_1 varies

is best linear approximation to f near x_0
<u>aka</u> tan line @ x_0

<u>Idea</u> given $f(x,y)$ and (x_0, y_0) in its domain, then the tan plane should be given by the linear fcn of x & y that best approx. $f(x,y)$ near (x_0, y_0)
Say (x_1, y_1) is near (x_0, y_0).

$$f(x_1, y_1) \approx f(x_0, y_0) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}$$

$$\approx \underbrace{f(x_0, y_0) + (x_1 - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y_1 - y_0)\frac{\partial f}{\partial y}(x_0, y_0)}_{\text{this is a linear fcn of } x_1 \text{ \& } y_1}$$

the Func

$$z = f(x_0, y_0) + (x_1 - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y_1 - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$$

is a linear fcn of x_1, y_1 (aka x, y) that is a good approx to $f(x_1, y_1)$ when (x_1, y_1) is near (x_0, y_0)

<u>Notice</u> $z = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$

def's a plane in $\mathbb{R}^3$

↳ It's the tan plane to $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$

find tangent plane at $(x_0, y_0) = (1, 2)$

<u>Recall</u> $\frac{\partial f}{\partial x} = ye^{xy}$

$\frac{\partial f}{\partial}$ ^{xy}

$$\frac{\partial f}{\partial x} = \ldots$$

$$\frac{\partial f}{\partial y} = x e^{xy}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(1,2)$$

$$= 2e^{(1)(2)} = 2e^2$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(1,2) = e^2$$

$$f(x_0, y_0) = e^2$$

$$\left.\begin{aligned} z &= e^2 + (x-1)(2e^2) + (y-2)e^2 \\ &= 2e^2 x + e^2 y + e^2 - 2e^2 - 2e^2 \\ &= 2e^2 x + e^2 y - 3e^2 \\ &= e^2(2x + y - 3) \end{aligned}\right\} \text{eq. of the tangent plane}$$